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HIGH-FREQUENCY LONG-WAVE SHELL VIBRATION*

V. L. BERDICHEVSKII and LE KHAN' CHAU

Equations describing high-frequency long-wave shell vibrations are constructed. They are an extension of the equations obtained earlier for plates, to shells /1/. The corresponding extension for straight rods is given in /2/.

1. Long-wave vibrations. A long-wave state is understood to be the state of stress whose characteristic scale change along the longitudinal coordinates l is considerably greater than the shell thickness h. The possible types of long-wave vibrations can be characterized qualitatively as follows. Let the face surfaces of the shell be load-free. Since $l \gg h$, the derivatives of the displacements with respect to the longitudinal coordinates ξ^α (the small Greek indices correspond to projections on the axis ξ^α and run through the values 1,2) can be neglected in the Lamé equations for the displacements and in the boundary conditions as compared with the derivatives with respect to the transverse coordinate ξ ($|\xi| \leqslant h/2$). Then the Lamé equations decompose into a system of three independent equations

$$\mu \frac{\partial^2 w_\alpha}{\partial \xi^2} = \rho \frac{\partial^2 w_\alpha}{\partial t^2}, \ |\xi| \leqslant \frac{h}{2}; \quad \frac{\partial w_\alpha}{\partial \xi} = 0, \ \xi = \pm \frac{h}{2}, \quad (\lambda + 2\mu) \frac{\partial^2 w}{\partial \xi^2} = \rho \frac{\partial^2 w}{\partial t^2}, \ |\xi| \leqslant \frac{h}{2}; \quad \frac{\partial w}{\partial \xi} = 0, \ \xi = \pm \frac{h}{2} \tag{1.1}$$

Here w_α and w are projections of the displacement onto the tangent vector and the normal to the middle surface.

Let us list the complete set of particular solutions of (1.1):

$$w = u \cos \alpha\zeta, \quad w_\sigma = 0, \quad \alpha = \pi n, \quad \zeta = 2\xi / h \qquad (F_\perp(n)) \tag{1.2}$$

$$w = 0, \quad w_\sigma = \psi_\sigma \sin \beta\zeta, \quad \beta = {}^1/_2\pi (2n + 1) \qquad (F_\parallel(n))$$

$$w = \psi \sin \alpha\zeta, \quad w_\sigma = 0, \quad \alpha = {}^1/_2\pi (2n + 1) \qquad (L_\perp(n))$$

$$w = 0, \quad w_\sigma = u_\sigma \cos \beta\zeta, \quad \beta = \pi n \qquad (L_\parallel(n))$$

The quantities α and β run through a countable number of values, however, no indices are superposed on α and β in order to avoid complicated notations. It is understood that its functions u, ψ_σ, ψ and u_σ correspond each value of α and β; these functions are also not numbered. In each particular solution the u, ψ_σ, ψ, and u_σ depend arbitrarily on the longitudinal coordinates and depend harmonically on t with frequency ω which is determined by the appropriate values of α and β from the formulas

$$\omega = 2\alpha c_1 / h \quad \text{or} \quad \omega = 2\beta c_2 / h, \quad (c_1^2 = (\lambda + 2\mu)/\rho, \quad c_2^2 = \mu/\rho, \quad \alpha = e\beta, \quad e = c_2 / c_1)$$

The notation of the appropriate solutions is indicated in parentheses in (1.2).

For functions $u, \psi_\sigma, \psi, u_\sigma$ independent of ξ^α, each of the solutions (1.2) represents an exact solution of the Lamé equations for an infinite plate and corresponds to vibrations of transverse fibers occurring in synchronization along the plate.

For vibrations whose amplitude and frequency vary slowly along the plate, as well as for shell vibrations, the equations (1.1) are zero approximations and the solutions (1.2) can be considered as the principal terms in a certain asymptotic expansion in which $u, \psi_\sigma, \psi, u_\sigma$ are functions of the longitudinal coordinates ξ^α and the time t, where

$$\partial u / \partial t \sim \omega u, \ \partial \psi_\sigma / \partial t \sim \omega \psi_\sigma, \ \partial \psi / \partial t \sim \omega \psi, \ \partial u_\sigma / \partial t \sim \omega u_\sigma$$

The values of ω in these estimates are taken for the same branch as the corresponding function, with the exception of $F_\perp(0)$ and $L_\parallel(0)$, for which it is assumed that $u_{,t} = O(c_1 u / l)$, $u_{\alpha,t} = O(c_1 u_\alpha / l)$, where l is the characteristic scale of the deformation /3/.

The branches $F_\perp(0)$ and $L_\parallel(0)$ correspond to the zero value of the natural vibrations frequency of a transverse fiber and corespond to the low-frequency vibrations when $\omega h / c_1 \ll 1$. Independence of the displacements at these branches from the transverse coordinate in the zero approximation is part of the Kirchhoff—Love hypothesis /4,5/. All the remaining branches correspond to vibrations with frequency $\omega \sim c_1 / h$. The propagation time for a perturbation over the thickness is commensurate with the period of vibration for them, and it is impossible to consider the displacements polynomials in the transverse coordinate even in a first approximation. Since $\omega \to \infty$ as $h \to 0$, the corresponding vibrations are naturally called high-

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frequency.

For instance, for $n=1, c_2=2500$ m/s, $h=1$ mm, we have $\omega_1 \approx 4\cdot 10^5$ Hz for the lowest "high" frequency, i.e., ω_1 is in the ultrasonic domain. Vibrations of elastic bodies at such a frequency can be substantial in problems on impact or in problems on vibrations caused by an electromagnetic field. Let us note that for shells inhomogeneous over the thickness and with a significant drop in the elastic moduli, ω_1 is considerably less and can even be in the audio frequency domain.

The branch $F_{\|}(0)$ corresponds to vibrations for which a shift of the transverse fiber into the half-wave of a sinusoid occurs. Attempts are made to take this kind of vibrations into account in Timoshenko-type shell theories. However, utilization of a linear displacement distribution over the thickness instead of the correct sinusoid in Timoshenko-type shell theories does not permit achieving a satisfactory quantitative correspondence.

The following vibrations modes oscillate all the more rapidly over the transverse coordinate as n grows. The branch numbered n has $2n$ or $2n+1$ nodes.

Any long-wave vibration can be represented as the sum of vibrations corresponding to different branches. Two-dimensional equations will be written below for each branch, by using a variational-asymptotic method /3/.

It turns out that the branches possess a remarkable property: they are orthogonal in the elastic and kinetic energies in a first approximation. This means that in a first approximation vibrations of one type do not cause vibrations of another type, and the possibility appears for investigating the vibrations of one branch independently of the vibrations of the rest.

2. Dependence of the displacement on the transverse coordinates.

We refer the undeformed state of the shell to the Lagrange curvilinear coordinates ξ^α, ξ

$$x^i = r^i(\xi^\alpha) + \xi n^i(\xi^\alpha), \quad -h/2 \leqslant \xi \leqslant h/2$$

Here x^i are Cartesian coordinates of the observer, $x^i = r^i(\xi^\alpha)$ is the location of the middle surface Ω, $n^i(\xi^\alpha)$ is the normal to Ω, h is the shell thickness, the Latin superscripts correspond to projections on the axes x^i and run through the values 1,2,3. In the Lagrange coordinate system the metric tensor and strain tensor components are given by the formulas

$$g_{\alpha\beta} = a_{\alpha\beta} - 2b_{\alpha\beta}\xi + c_{\alpha\beta}\xi^2, \quad g_{\alpha 3}=0, \quad g_{33}=1, \quad g^{\alpha\beta} = a^{\alpha\beta} + 2b^{\alpha\beta}\xi + 3c^{\alpha\beta}\xi^2 + O(h^3/R^3); \quad g^{\alpha 3}=0, \quad g^{33}=1 \quad (2.1)$$

$$\varepsilon_{\alpha\beta} = x^i_{(,\alpha} w_{i,\beta)} = r^i_{(\alpha} w_{i,\beta)} - \xi b^\sigma_{(\alpha} r_\sigma{}^i w_{i,\beta)} = w_{(\alpha;\beta)} - w b_{\alpha\beta} - \xi b^\sigma_{(\alpha} w_{\sigma;\beta)} + \xi w c_{\alpha\beta}$$

$$2\varepsilon_{3\alpha} = n^i w_{i,\alpha} + x^i_{,\alpha} w_{i,\xi} = w_{,\alpha} + b_\alpha{}^\sigma w_\sigma + w_{\alpha,\xi} - b_\alpha{}^\sigma \xi w_{\sigma,\xi}, \quad \varepsilon_{33} = n^i w_{i,\xi} = w_{,\xi}$$

Here $w_\alpha^i = w_i r_\alpha{}^i$, $w = w_i n^i$ are the projections of the displacement vector w_i on the tangent vectors $r_\alpha{}^i \equiv r^i{}_{,\alpha}$ and the normal n^i, the subscript 3 corresponds to the projection on the normal n^i, $a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}$ are, respectively, the first, second and third quadratic forms of the middle surface Ω, the comma in the subscripts denotes partial differentiation with respect to the ξ^α, the semicolon denotes covariant differentiation relative to the metric $a_{\alpha\beta}$, and the parentheses in the subscripts denote the symmetrization operation.

Let the shell edge be rigidly fixed

$$w_i = 0 \quad \text{on } \Gamma \times [-h/2, h/2] \quad (2.2)$$

where Γ is the boundary of the middle surface Ω. Let us first set the external load on the face surfaces P^i equal to zero. Then the displacements corresponding to free vibrations of the shell are extremals of the functional

$$I = \int_{t_1}^{t_2}\int_\Omega \int_{-h/2}^{h/2} (U - K)\varkappa \, d\xi \, d\Omega \, dt \quad (2.3)$$

$$2U = \lambda (g^{\alpha\beta}\varepsilon_{\alpha\beta} + \varepsilon_{33})^2 + 2\mu g^{\alpha\gamma} g^{\beta\delta} \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + 2\mu\varepsilon_{33}{}^2 + 4\mu g^{\alpha\beta}\varepsilon_{\alpha 3}\varepsilon_{\beta 3}, \quad 2K = \rho w_{,t}{}^2 + \rho a^{\alpha\beta} w_{\alpha,t} w_{\beta,t}, \quad \varkappa = 1 - 2H\xi + K\xi^2$$

Here H and K are the median and Gaussian curvature of Ω, and $d\Omega$ is an area element on Ω.

Let us investigate the functional (2.3) by using the variational-asymptotic method /3,6/.

We assume that the dimensionless parameters $h_* = h/R$ (R is the shell minimum radius of curvature) and $h_{**} = h/l$ are small everywhere in Ω. We make the substitution $\zeta = 2\xi/h$ and we discard all small terms in the asymptotic sense. We arrive at the zero approximation functional

$$2I = \frac{h}{2}\int_{t_1}^{t_2}\int_\Omega \int_{-1}^{1} \left[(\lambda+2\mu)\frac{4}{h^2} w_{,\zeta}^2 + \mu a^{\alpha\beta}\frac{4}{h^2} w_{\alpha,\zeta} w_{\beta,\zeta} - \rho w_{,t}^2 - \rho a^{\alpha\beta} w_{\alpha,t} w_{\beta,t}\right] d\zeta \, d\Omega \, dt$$

The Euler equations of this functional yield four series of natural vibrations (1.2).

We impose the constraints

$$u = \psi_\sigma = \psi = u_\sigma = 0 \ \text{ on } \ \Gamma \tag{2.4}$$

on the functions $u, \psi_\sigma, \psi, u_\sigma$ to satisfy the rigid fixing conditions (2.2). Now we find the next approximation for the displacement of branches of the series $F_\perp$. Considering u a given function of ξ^α, we seek w_α. Keeping the principle terms dependent on w_α and the principal cross terms in (2.3), we obtain the functional

$$I = \frac{h}{2}\int_{t_1}^{t_2}\int_\Omega \bar\Lambda \, d\Omega \, dt \tag{2.5}$$

$$2\bar\Lambda = \langle 2\lambda w^\sigma u_{,\sigma} 2\alpha h^{-1}\sin\alpha\zeta + \mu a^{\lambda\sigma}(2h^{-1}w_{\lambda,\zeta} + u_{,\lambda}\cos\alpha\zeta)(2h^{-1}w_{\sigma,\zeta} + u_{,\sigma}\cos\alpha\zeta) - \rho\omega^2 w_\sigma w^\sigma\rangle$$

The integral with respect to ζ in the limits /-1,1/ is denoted by $\langle\cdot\rangle$. Integration by parts was performed and the boundary conditions (2.4) were used in the derivation of (2.5). Let us find the extremal of this functional. After taking the variation of (2.5) relative to w_α we obtain the equations

$$w_{\sigma,\zeta\zeta} + \beta^2 w_\sigma = [(\lambda+\mu)/(2\mu)]\, h\alpha u_{,\sigma}\sin\alpha\zeta, \ |\zeta|\leqslant 1, \quad w_{\sigma,\zeta} + {}^1/_2\, h u_{,\sigma}\cos\alpha\zeta = 0, \ \zeta = \pm 1$$

They result in the following value of the tangential displacements

$$w_\sigma = u_{,\sigma}\frac{h}{2\alpha}\left(\sin\alpha\zeta - \frac{2(-1)^n e\sin\beta\zeta}{\cos\beta}\right) \tag{2.6}$$

As should have been expected, the tangential displacements turned out to be very much less than the normal displacement ($w_\sigma = 0$ in the zero approximation), and are on the order of $h_{**}u$.

Let us seek the correction to w

$$w = u\cos\alpha\zeta + w'$$

Here w_0 is considered fixed and defined by (2.6).

Without limiting the generality, the following constraint can be imposed on w':

$$\langle w'\cos\alpha\zeta\rangle = 0$$

It corresponds to the assumption that $u = \langle w\cos\alpha\zeta\rangle$.

After discarding small terms containing w' and small cross terms as compared with the rest, the functional (2.3) takes the form (2.5) with a Lagrangian given by the formula

$$\bar\Lambda' = \Big\langle \frac{4}{h^2}(\lambda+2\mu)\, w'^2_{,\zeta} + \frac{2(\lambda+2\mu)}{h}4H\zeta\alpha u\sin\alpha\zeta w'_{,\zeta} - 2\lambda\frac{4H}{h}w'_{,\zeta}u\cos\alpha\zeta + 2\lambda\frac{4H}{h}w'\alpha\sin\alpha\zeta - \rho(w'_{,t})^2 - 2\rho Hh\zeta u_{,t}\cos\alpha\zeta\, w'_{,t}\Big\rangle$$

Its extremal has the form

$$w' = \frac{Hh}{2}u\left(\zeta\cos\alpha\zeta + \frac{1-4e^2}{\alpha}\sin\alpha\zeta\right)$$

Finally, we have the following distribution of the displacements over the thickness in the series $F_\perp$ (to the accuracy of second order terms in h_* and h_{**}):

$$F_\perp:\ w = u\cos\alpha\zeta + \frac{Hh}{2}u\left(\zeta\cos\alpha\zeta + \frac{1-4e^2}{\alpha}\sin\alpha\zeta\right), \quad w_\sigma = u_{,\sigma}\frac{h}{2\alpha}\left(\sin\alpha\zeta - \frac{2(-1)^n e\sin\beta\zeta}{\cos\beta}\right), \quad \alpha\neq 0 \tag{2.7}$$

Formulas are obtained analogously for the displacements in the three remaining series

$$F_\parallel:\ w_\sigma = \psi_\sigma\sin\beta\zeta + \frac{h}{2}\left(H\psi_\sigma\zeta\sin\beta\zeta + \frac{\bar b_\sigma^{\ \lambda}}{\beta}\psi_\lambda\cos\beta\zeta\right), \qquad w = \psi^\sigma_{;\sigma}\frac{h}{2\beta}\left(\cos\beta\zeta - \frac{2(-1)^n e\cos\alpha\zeta}{\sin\alpha}\right) \tag{2.8}$$

$$L_\perp:\ w = \psi\sin\alpha\zeta + \frac{Hh}{2}\psi\left(\zeta\sin\alpha\zeta - \frac{1-4e^2}{\alpha}\cos\alpha\zeta\right), \quad w_\sigma = \psi_{,\sigma}\frac{h}{2\alpha}\left(-\cos\alpha\zeta + \frac{2(-1)^n e\cos\beta\zeta}{\sin\beta}\right)$$

$$L_\parallel:\ w_\sigma = u_\sigma\cos\beta\zeta + \frac{h}{2}\left(Hu_\sigma\zeta\cos\beta\zeta - \frac{\bar b_\sigma^{\ \lambda}}{\beta}u_\lambda\sin\beta\zeta\right)$$

$$w = u^\sigma_{;\sigma}\frac{h}{2\beta}\left(-\sin\beta\zeta + \frac{2(-1)^n e\sin\alpha\zeta}{\cos\alpha}\right), \quad \beta\neq 0, \quad (\bar b_\alpha^{\ \beta} = b_\alpha^{\ \beta} + H\delta_\alpha^{\ \beta})$$

The fundamental shell feature as compared with plates is that the correction terms in the displacements are of the order h_* compared to the principal term, while they are of order h_{**}^2 in plates.

By continuing the iteration process, the next corrections to w and w_α can be found. They are not written down here since they yield no contribution to the average Lagrangian of the first approximation.

3. Average Lagrangians. Let us assume that the quantities $u, \psi_\sigma, \psi, u_\sigma$ in the formulas for the displacements are arbitrary functions of ξ^α and t. Substituting these formulas into the functional (2.3), and retaining components of order h_*^{-2}, h_{**}^{-2} and 1 as compared with one, we obtain functionals of the type (2.5) with the following average Lagrangians

$$F_\perp:\ 2\bar\Lambda = (\lambda + 2\mu)\left(\left(\frac{2\alpha}{h}\right)^2 u^2 + k_1 a^{\lambda\sigma} u_{\cdot\lambda} u_{\cdot\sigma} + k_3 u^2\right) - \rho u_{\cdot t}^2 - \rho a^{\lambda\sigma} u_{\cdot\lambda t} u_{\cdot\sigma t}\left(\frac{h}{2\alpha}\right)^2 k_2 - \rho u_{\cdot t}^2 \left(\frac{h}{2\alpha}\right)^2 k_4, \quad \alpha \neq 0 \tag{3.1}$$

$$k_1 = 2\left(1 - \frac{2e^2 \operatorname{tg}\beta}{\beta}\frac{5-3e^2}{1-e^2} + \frac{2e^2}{\cos^2\beta}\right)$$

$$k_2 = 1 - \frac{3-e^2}{1-e^2}\frac{4e^2\operatorname{tg}\beta}{\beta} + \frac{4e^2}{\cos^2\beta}$$

$$k_3 = -(3H^2 - K)\left(\frac{3}{2} + \frac{\alpha^2}{3} - 8e^2\right) - 4H^2(1 - 3e^2 + 4e^4)$$

$$k_4 = -(3H^2 - K)\left(\frac{1}{2} + \frac{\alpha^2}{3}\right) + 2H^2(1 - 6e^2 + 8e^4)$$

$$F_\parallel:\ 2\bar\Lambda = \mu\left(\left(\frac{2\beta}{h}\right)^2 a^{\sigma\lambda} + k_3^{\sigma\lambda}\right)\psi_\sigma\psi_\lambda + 2\mu\psi_{(\alpha;\beta)}\psi^{(\beta;\alpha)} + (\lambda + 2\mu)k_1(\psi^\sigma_{;\sigma})^2 - \rho\left(a^{\sigma\lambda} + \left(\frac{h}{2\beta}\right)^2 k_4^{\sigma\lambda}\right)\psi_{\sigma,t}\psi_{\lambda,t} - \rho\left(\frac{h}{2\beta}\right)^2 k_2(\psi^\sigma_{;\sigma t})^2$$

$$k_1 = \frac{4e^4}{\sin^2\alpha}\left(1 + \frac{1-3e^2}{1-e^2}\frac{\sin 2\alpha}{2\alpha}\right)$$

$$k_2 = 1 + \frac{4e^2}{\sin^2\alpha}\left(1 - \frac{1+e^2}{1-e^2}\frac{\sin 2\alpha}{2\alpha}\right)$$

$$k_3^{\alpha\beta} = 6Hb^{\alpha\beta} - ((3H^2 - K)a^{\alpha\beta} - 2b^{\alpha\beta}H)\left(\frac{\beta^2}{3} - \frac{1}{2}\right)$$

$$k_4^{\alpha\beta} = 2((3Hb^{\alpha\beta}) - Ka^{\alpha\beta}) - ((3H^2 - K)a^{\alpha\beta} - 2Hb^{\alpha\beta})\left(\frac{\beta^2}{3} + \frac{1}{2}\right)$$

$$L_\perp:\ 2\bar\Lambda = (\lambda + 2\mu)\left(\left(\frac{2\alpha}{h}\right)^2\psi^2 + k_1 a^{\lambda\sigma}\psi_{\cdot\lambda}\psi_{\cdot\sigma} + k_3\psi^2\right) - \rho\psi_{\cdot t}^2 - \rho a^{\lambda\sigma}\psi_{\cdot\lambda t}\psi_{\cdot\sigma t}\left(\frac{h}{2\alpha}\right)^2 k_2 - \rho\psi_{\cdot t}^2\left(\frac{h}{2\alpha}\right)^2 k_4$$

$$k_1 = 2\left(1 + \frac{2e^2\operatorname{ctg}\beta}{\beta}\frac{5-3e^2}{1-e^2} + \frac{2e^2}{\sin^2\beta}\right)$$

$$k_2 = 1 + \frac{3-e^2}{1-e^2}\frac{4e^2\operatorname{ctg}\beta}{\beta} + \frac{4e^2}{\sin^2\beta}$$

$$L_\parallel:\ 2\bar\Lambda = \mu\left(\left(\frac{2\beta}{h}\right)^2 a^{\sigma\lambda} + k_3^{\sigma\lambda}\right)u_\sigma u_\lambda + 2\mu u_{(\alpha;\beta)}u^{(\beta;\alpha)} + (\lambda + 2\mu)k_1(u^\sigma_{;\sigma})^2 - \rho\left(a^{\sigma\lambda} + \left(\frac{h}{2\beta}\right)^2 k_4^{\sigma\lambda}\right)u_{\sigma,t}u_{\lambda,t} - \rho\left(\frac{h}{2\beta}\right)^2 k_2(u^\sigma_{;\sigma t})^2$$

$$k_1 = \frac{4e^4}{\cos^2\alpha}\left(1 - \frac{1-3e^2}{1-e^2}\frac{\sin 2\alpha}{2\alpha}\right)$$

$$k_2 = 1 + \frac{4e^2}{\cos^2\alpha}\left(1 + \frac{1+e^2}{1-e^2}\frac{\sin 2\alpha}{2\alpha}\right)$$

Expressions for the coefficients k_3, k_4 in the series $L_\perp$, and for the tensors $k_3^{\alpha\beta}$, $k_4^{\alpha\beta}$ in the series $L_\parallel$ are not written down here since they agree in form with the corresponding expressions in the series $F_\perp$ and $F_\parallel$. It is considered that α, β in the series $F_\perp$ and $L_\parallel$ are different from zero. The branches $F_\perp(0)$ and $L_\parallel(0)$ $(\alpha, \beta \to 0)$ correspond to classical theory and the appropriate Lagrangians are not written down.

The coefficients with subscripts 1 and 2 agree with the coefficients so denoted for the plates /1/, and the formulas go over into the corresponding formulas for plates when the coefficients with subscripts 3 and 4 vanish.

Not only the "principal" terms (containing the factor h^{-2} and the differentiation with respect to t) must be retained in the average Lagrangians, but also terms of the next order of smallness, which is related to the fact that the sum of the principal terms at the "natural" frequencies ω turns out to be small.

4. Orthogonality in energy. It turns out that, just as for plates /1/, different branches of shell vibrations are orthogonal relative to the elastic and kinetic energies to an error not less than $h_*^2 + h_{**}^2 + h_* h_{**}$ as compared with one, i.e.,

$$\int_\Omega \langle\sigma_1^{ij}\varepsilon_{2ij}\varkappa\rangle\, d\Omega = 0, \quad \int_\Omega \langle\rho w^i_{1,t} w_{2i,t}\rangle\, d\Omega = 0 \tag{4.1}$$

The exception is the classical vibrations, the branches $F_\perp(0)$ and $L_\parallel(0)$ which are not mutually orthogonal but are orthogonal to all the other branches.

The property (4.1) is satisfied under the condition of rigid support (2.5). The components of order h_*^2 and h_{**}^2 in the expressions for the displacements must be taking into account to prove (4.1).

The proof is by direct substitution of expressions for the displacement into the functional; the following assertion is established first: any two branches satisfy the following orthogonality condition independently of the boundary conditions (to the accuracy of terms on the order of $h_*^2 + h_{**}^2 + h_* h_{**}$ as compared with one)

$$\int_{t_1}^{t_2}\int_{\Omega} \langle \varkappa (\sigma_{(1}^{ij}\varepsilon_{2)ij} - \rho w_{(1,\, t}^{i} w_{2)\, i,\, t} \rangle \, d\Omega \, dt = 0 \tag{4.2}$$

As for (4.1), the exception for (4.2) is given by the branches $F_{\perp}(0)$ and $L_{\parallel}(0)$. The orthogonality relative to the kinetic energy is then confirmed (by using the rigid support conditions). The appropriate calculations are not presented because of their awkwardness.

5. Equations of forced shell vibrations. Now we assume that external surface forces $P_{\pm}^{i}$ are applied to the face surfaces of the shell (at $\xi = \pm h/2$). The solution of the problem is the extremal of the functional (U, K are defined by formulas (2.3))

$$I = \frac{h}{2}\int_{t_1}^{t_2}\int_{\Omega}\int_{-1}^{1} \Lambda \, d\zeta \, d\Omega \, dt - \int_{t_1}^{t_2}\int_{\Omega} \{\varkappa P^i w_i\} \, d\Omega \, dt, \quad \Lambda = \varkappa (U - \mathrm{K}), \quad \{A\} \equiv A_{\xi = h/2} + A\,|_{\xi = -h/2} \tag{5.1}$$

We seek the solution of the problem in the form

$$w \equiv w^i n_i = \sum_{n=1}^{\infty} w_{F_{\perp}(n)} + w_{F_{\perp}(n)} + w_{L_{\perp}(n)} + w_{L_{\parallel}(n)}) + w_0 \tag{5.2}$$

$$w^{\alpha} \equiv w^i r_i^{\alpha} = \sum_{n=1}^{\infty} (w_{F_{\perp}(n)}^{\alpha} + w_{F_{\parallel}(n)}^{\alpha} + w_{L_{\perp}(n)}^{\alpha} + w_{L_{\parallel}(n)}^{\alpha}) + w_0^{\alpha}$$

The displacements of the natural branches are here expanded to second order terms in h (h_*^2, h_{**}^2 and $h_* h_{**}$), and the functions $u, \psi_{\alpha}, \psi, u_{\alpha}$ in these expansions are arbitrary functions of ξ^{α} and t. The classical displacements w_0 and w_0^{α} are extracted specially from the series $F_{\perp}$ and $L_{\parallel}$. It is understood that the characteristic scale of variation of the external force P_i is very much greater than the shell thickness, and $P_i = O(\mu h_{**}\varepsilon)$ (ε is the strain amplitude of the shell).

Let us substitute (5.2) into the functional (5.1) and retain terms on the order of h_*^{-2}, h_*^{-1} h_{**}^{-1}, h_{**}^{-2} and 1. By using the orthogonality of the different branches we obtain that the Lagrangian consists of the classical Lagrangian of the low-frequency vibrations and the following Lagrangians of the high-frequency vibrations:

$$F_{\perp}\colon \Lambda^{(P)} = \frac{h}{2}\overline{\Lambda} - \{P\}(-1)^n u + \frac{hH}{2}[P](-1)^n u + [P^{\sigma}]\frac{he(-1)^n \operatorname{tg}\beta}{\alpha} u_{,\sigma} \tag{5.3}$$

$$F_{\parallel}\colon \Lambda^{(P)} = \frac{h}{2}\overline{\Lambda} - [P^{\alpha}](-1)^n \psi_{\alpha} + \frac{hH}{2}\{P^{\alpha}\}(-1)^n \psi_{\alpha} + \{P\}\frac{he(-1)^n \operatorname{ctg}\alpha}{\beta}\psi^{\sigma}{}_{;\sigma}$$

$$L_{\perp}\colon \Lambda^{(P)} = \frac{h}{2}\overline{\Lambda} - [P](-1)^n \psi + \frac{hH}{2}\{P\}(-1)^n \psi - \{P^{\sigma}\}\frac{he(-1)^n \operatorname{ctg}\beta}{\alpha}\psi_{,\sigma}$$

$$L_{\parallel}\colon \Lambda^{(P)} = \frac{h}{2}\overline{\Lambda} - \{P^{\alpha}\}(-1)^n u_{\alpha} + \frac{hH}{2}[P^{\alpha}](-1)^n u_{\alpha} - [P]\frac{he(-1)^n \operatorname{tg}\alpha}{\beta}u^{\sigma}_{;\sigma}$$

$$P = P^i n_i, \quad P_{\alpha} = P_i r_{\alpha}^i, \quad [A] = A\,|_{\xi = h/2} - A\,|_{\xi = -h/2}$$

Here $\overline{\Lambda}$ are defined by (3.1).

By varying the action, we obtain the forced high-frequency vibrations equations

$$F_{\perp}\colon \frac{h}{2}(\lambda + 2\mu)\left[\left(\left(\frac{2\alpha}{h}\right)^2 + k_3\right)u - k_1 \Delta u\right] + \rho\frac{h}{2}\left[1 + \left(\frac{h}{2\alpha}\right)^2 k_4\right]u_{,tt} - \rho\frac{h}{2}\left(\frac{h}{2\alpha}\right)^2 k_2 \Delta u_{,tt} =$$

$$\{P\}(-1)^n - \frac{hH}{2}[P](-1)^n + [P_{;\alpha}^{\alpha}]\frac{he(-1)^n \operatorname{tg}\beta}{\alpha}$$

$$F_{\parallel}\colon \frac{\mu h}{2}\left[\left(\frac{2\beta}{h}\right)^2 \delta_{\alpha}^{\sigma} + k_{3\alpha}^{\sigma}\right]\psi_{\sigma} + \frac{\rho h}{2}\left[\delta_{\alpha}^{\sigma} + \left(\frac{h}{2\beta}\right)^2 k_{4\alpha}^{\sigma}\right]\psi_{\sigma,\, tt}$$

$$-\frac{h}{2}[(\lambda + 2\mu)k_1 + \mu]\psi^{\sigma}_{;\sigma\alpha} - \frac{\mu h}{2}\Delta\psi_{\alpha} - \frac{\rho h}{2}\left(\frac{h}{2\beta}\right)^2 k_2 \psi^{\sigma}_{;\sigma\alpha tt} = [P_{\alpha}](-1)^n - \frac{hH}{2}\{P_{\alpha}\}(-1)^n + \{P_{,\alpha}\}\frac{he(-1)^n \operatorname{tg}\alpha}{\beta}$$

The equations for the branches $L_{\perp}$ and $L_{\parallel}$ agree with the equations for the branches $F_{\perp}$ and $F_{\parallel}$ upon making the respective substitutions: $u \to \psi$, $\operatorname{tg}\beta \to (-\operatorname{ctg}\beta)$ and $\psi_{\sigma} \to u_{\sigma}$, $\operatorname{ctg}\alpha \to (-\operatorname{tg}\alpha)$.

REFERENCES

1. BERDICHEVSKII V.L., High-frequency long-wave plate vibrations, Dokl. Akad. Nauk SSSR, Vol. 236, No.6, 1977.

2. KVASHNINA S.S., High-frequency long-wave vibrations of elastic rods, PMM, Vol.43, No.2, 1979.

3. BERDICHEVSKII V.L., Variational-asymptotic method of constructing a theory of shells. PMM Vol.43, No.4, 1979.

4. GOL'DENVEIZER A.L., Theory of Thin Elastic Shells. English translation, Pergamon Press, Book No. 09561, 1961.

5. NOVOZHILOV V.V., Theory of Thin Shells. Sudpromgiz, Leningrad, 1964.

6. BERDICHEVSKII V.L., Variational-asymptotic method, IN: Certain Questions of the Mechanics of Continuous Media. Moscow State University Press, Moscow, 1978.

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